

Anomalous dimensions of operators in polarized deep inelastic scattering at $O(1/N_f)$

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Abstract. Critical exponents are computed for a variety of twist-2 composite operators, which occur in polarized and unpolarized deep inelastic scattering, at leading order in the $1/N_f$ expansion. The resulting d -dimensional expressions, which depend on the moment of the operator, are in agreement with recent explicit two and three loop perturbative calculations. An interesting aspect of the critical point approach which is used, is that the anomalous dimensions of the flavour singlet eigenoperators, which diagonalize the perturbative mixing matrix, are computed directly. We also elucidate the treatment of γ^5 at the fixed point which is important in simplifying the calculation for polarized operators. Finally, the anomalous dimension of the singlet axial current is determined at $O(1/N_f)$ by considering the renormalization of the anomaly in operator form.

1 Introduction.

Our understanding of the structure of nucleons is derived primarily from experiments where they are bombarded by other nucleons or electrons at high energies. These deeply inelastic processes are, in general, well understood in most instances. Current activity, however, centres on examining polarized reactions due, for example, to the discrepancy observed in results for the spin of the proton and theoretical predictions by the EMC collaboration, [1]. Consequently in order to make accurate statements about the data current theoretical interest has focussed on carrying out higher order perturbative calculations in the underlying field theory, quantum chromodynamics, (QCD). As this theory is asymptotically free at high energies, [2], the coupling constant is sufficiently small so that perturbative calculations give a good description of the deep inelastic phenomenology. Indeed unpolarized scattering is well understood with one and two loop results available for the anomalous dimensions of the twist-2 flavour non-singlet and singlet operators which arise in the operator product expansion, [3-5]. Moreover, the Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, (DGLAP), splitting functions, [6], which are a measure of the probability that a constituent parton fragments into other partons are known to the same accuracy, [3, 5]. The moments of the scattering amplitudes have also been studied. More recently the three loop structure has been obtained exactly, for low moments, through huge impressive analytic computations, both for non-singlet and singlet cases, [7].

The situation for polarized scattering is less well established. The one loop anomalous dimensions for the corresponding twist-2 (and 3) operators were computed by Ahmed and Ross in [8]. However, only recently has the two loop structure been determined for the flavour singlet operators, [9]. (The non-singlet polarized dimensions are equivalent to the non-singlet unpolarized case.) This was checked by Vogelsang in [10, 11] by calculating the splitting functions themselves and then comparing with [9] by taking the inverse Mellin transform. In this the moment n of the operator has the conjugate variable x which is the momentum fraction of the parton in the nucleon. These results have been important for the next to leading order evolution of the structure functions to low x and Q^2 regions, [12]. (For review articles see [13].) Crucial in this exercise is the dependence of the results on the moment of the operators.

To go beyond this two loop picture would require a great amount of computation based, for example, on the unpolarized results of [7]. One way of improving our knowledge would be to compute the appropriate quantities using another approximation. For example, the large N_f expansion, where N_f is the number of quark flavours, has been used to obtain the leading order coefficients of the anomalous dimensions of the twist-2 unpolarized non-singlet operators to all orders in the perturbative coupling constant, [14]. The resulting analytic function of n provided a useful check on the exact low moment non-singlet three loop calculation of [7]. Briefly the method involved studying the scaling behaviour of the operator at the non-trivial fixed point in d -dimensional QCD. With that success and the current interest in polarized physics required for exploring new x régimes it is appropriate to apply the large N_f analysis to study the dimensions of the underlying (twist-2) operators. Aside from providing n -dependent results and getting a flavour of the structure beyond two loops, it will give at least another partial check on the recent results of [9-11]. In particular we will determine the critical exponents at $O(1/N_f)$ which encode the all orders coefficients of the twist-2 polarized singlet operators. As a prelude we need to study the unpolarized singlet case which will extend the result of [14].

The paper is organised as follows. The basic formalism and notation is introduced in section 2. We review previous work in section 3, including the technical details of the computation of singlet unpolarized operators anomalous dimensions. As the treatment of four dimensional objects, such as γ^5 , is needed we review previous $1/N_f$ work involving this in section 4. The

remaining sections 5 and 6 are devoted to the application of the results of earlier sections to polarized operators. In particular the latter section centers on the treatment of the singlet axial current which is not conserved due to the chiral anomaly. Future work and perspectives are discussed in section 7. An appendix gives details of the relation of the $1/N_f$ exponent results to the DGLAP splitting functions.

2 Background.

To begin with we review several of the more field theoretic aspects of the formalism including the role of the critical renormalization group. First, we recall that critical exponents are fundamental quantities. In experiments and condensed matter problems dealing with phase transitions they completely characterize the physics. In order to describe such phenomena one determines estimates of the exponents by calculating, for example, in the underlying quantum field theory describing the transition. In practice this means carrying out a perturbative calculation of the renormalization constants of the theory, in some renormalization scheme. This information is then encoded in the corresponding renormalization group functions such as the anomalous dimensions of the field or the mass. The critical coupling, g_c , is subsequently determined from the β -function of the theory. It is defined to be a non-trivial zero of $\beta(g)$. The appropriate critical exponents are then found by evaluating the anomalous dimensions at the critical coupling. In practice provided enough terms of the series have been computed, relatively accurate numerical estimates can be obtained. (Useful background material can be found in, for example, [15].)

For the present problem we will examine a fixed point in QCD in d -dimensions and obtain the critical exponents as a function of d and N_f which characterize the transition. As indicated these correspond to the renormalization group equation, (RGE), functions evaluated at g_c . Therefore provided the location of g_c is known in some approximation like $1/N_f$ one can decode the information contained in the exponent and determine the coefficients of the RGE functions for non-critical values of the coupling. In this large N_f method it turns out that the structure of $\beta(g)$ is such that for leading order calculations in $1/N_f$ only knowledge of the one loop coefficient is required, [16]. We illustrate these remarks with a general example, which will set notation for later sections. As we are interested in the coefficients in the series of an RGE function we denote such a function by $\gamma(g)$ and define its expansion to be, with explicit N_f dependence,

$$\gamma(g) = a_1 g + (a_2 N_f + b_1) g^2 + (a_3 N_f^2 + b_2 N_f + c_1) g^3 + O(g^4) \quad (2.1)$$

where the obvious definition for the $O(g^4)$ term is understood. The coefficients $\{a_i, b_i, c_i, \dots\}$ can of course be functions of other parameters such as colour group Casimirs or n . To evaluate (2.1) at a fixed point we take the general structure of the β -function to be, in d -dimensions,

$$\beta(g) = (d-4)g + AN_f g^2 + (BN_f + C)g^3 + O(g^4) \quad (2.2)$$

Setting $d = 4 - 2\epsilon$, then there is a non-trivial fixed point, g_c , at

$$g_c = \frac{2\epsilon}{AN_f} + O\left(\frac{1}{N_f^2}\right) \quad (2.3)$$

So at $O(1/N_f)$

$$\eta \equiv \gamma(g_c) = \frac{1}{N_f} \sum_{n=1}^{\infty} \frac{2^n a_n \epsilon^n}{A^n} + O\left(\frac{1}{N_f^2}\right) \quad (2.4)$$

where η is the corresponding critical exponent. Clearly at leading order in $1/N_f$, when N_f is large, the coefficients $\{a_i\}$ are accessed. To determine them one needs to compute η directly in

the $O(1/N_f)$ expansion in d -dimensions. This is the aim of the paper for the operators discussed earlier.

Before reviewing that we make several parenthetical remarks. In assuming a β -function of the form (2.2) we are restricting ourselves to a particular class of theories which includes QED and QCD. If the two loop term of (2.2) had been quadratic and not linear in N_f and likewise the three loop term cubic in N_f and so on, then it would not be possible to determine a simple form for g_c at leading order in $1/N_f$. Instead an infinite number of terms of $\beta(g)$ would be required. This would imply a large N_f expansion would not be possible in that case. This is similar to the large N_c expansion of QCD. Then the structure of the β -function has this nasty form and it is not easy to study QCD in a critical $1/N_c$ approach.

The method to compute critical exponents corresponding to the anomalous dimensions of operators in powers of $1/N_f$ is based on an impressive series of papers, [17-19]. In [16, 17] the $O(N)$ σ model was considered and the technique has been developed for fermion and gauge theories more recently, [20, 21]. Essentially one studies the theory precisely at the fixed point g_c where there are several simplifications. First, at g_c there is no mass in the problem so all propagators are massless. Second, the structure of the (full) propagators can be written down, in the approach to criticality. Therefore in momentum space a fermion and gauge field will have the respective propagators ψ and $A_{\mu\nu}$ of the form, in the limit $k^2 \rightarrow \infty$ [16],

$$\psi(k) \sim \frac{A k}{(k^2)^{\mu-\alpha}} \quad , \quad A_{\mu\nu}(k) \sim \frac{B}{(k^2)^{\mu-\beta}} \left[\eta_{\mu\nu} - (1-b) \frac{k_\mu k_\nu}{k^2} \right] \quad (2.5)$$

where $d = 2\mu$, b is the covariant gauge parameter and A and B are amplitudes. The dimensions α and β of the fields (in coordinate space) comprise two pieces. For example,

$$\alpha = \mu - 1 + \frac{1}{2}\eta \quad (2.6)$$

where the first term is the canonical dimension of the fermion determined by ensuring that the kinetic term in the action is dimensionless. The second term is the critical exponent corresponding to the anomalous dimension of ψ or the wave function renormalization evaluated at g_c . It reflects the effect radiative corrections have on the dimension of ψ .

With (2.5) one can analyse any set of Feynman diagrams in the neighbourhood of g_c and determine the scaling behaviour of the integral. In particular one can examine the 2-point Schwinger Dyson equation at criticality to obtain a representation of those equations. It turns out that one obtains a set of self-consistent equations which can be solved to determine η analytically as a function of d . Furthermore the approach is systematic in that $O(1/N_f^2)$ corrections can be studied too. In [19] this was extended to n -point Green's function. If, for example, one considers the 3-point interaction then the exponent or the vertex anomalous dimension is found by computing the (regularized) set of leading order integrals with (2.5). The residue of the simple pole of each graph contributes to the anomalous dimension. We will illustrate these remarks explicitly in the next section. However, we note that the regularization that is used is obtained by replacing β of (2.5) by $\beta - \Delta$. Here Δ is assumed to be small like the ϵ used in dimensional regularization, [19].

We conclude this section by recalling another feature of critical theory which is important in analysing QCD in large N_f . So far the above remarks have been completely general and summarize the approach taken in other models. Another common feature is that the theory that underlies a fixed point is not necessarily the unique model describing the physics. More than one model can be used to determine the (measured) critical exponents. In this case such theories are said to be in the same universality class. From a field theoretic point of view one can use this to simplify large N_f calculations. For example, the $O(N)$ σ model and ϕ^4 theory

with an $O(N)$ symmetry are equivalent at the d -dimensional fixed point where the former is defined in $2 + \epsilon$ dimensions and the latter in $4 - \epsilon$, [15]. The critical exponents computed in either are the same. For QCD there is also a similar equivalence which has been demonstrated by Hasenfratz and Hasenfratz in [12]. They showed that as $N_f \rightarrow \infty$ QCD and a non-abelian version of the Thirring model are equivalent. The lagrangians of each are, for QCD,

$$L = i\bar{\psi}^{iI}(\not{D}\psi)^{iI} - \frac{(G_{\mu\nu}^a)^2}{4e^2} \quad (2.7)$$

and

$$L = i\bar{\psi}^{iI}(\not{D}\psi)^{iI} - \frac{(A_\mu^a)^2}{2\lambda^2} \quad (2.8)$$

for the non-abelian Thirring model, (NATM), where $1 \leq i \leq N_f$, $1 \leq I \leq N_c$, $1 \leq a \leq N_c^2 - 1$, $D_{\mu IJ} = \partial_\mu \delta_{IJ} + T_{IJ}^a A_\mu^a$, $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$, ψ^{iI} is the quark field and A_μ^a is the gluon field. The coupling constants of each lagrangian are e and λ and are dimensionless in 4 and 2 dimensions respectively. The auxiliary spin-1 field of the NATM can be eliminated to produce a 4-fermi interaction which is renormalizable in strictly two dimensions. Essentially at a fixed point it is the interactions which are important and which suggest (2.7) and (2.8) are equivalent. The quadratic terms serve only to define the canonical dimensions of the fields and coupling constants as well as establishing various scaling laws for the exponents. Although the NATM does not appear to contain the triple and quartic vertices typical of a Yang Mills theory, it was demonstrated in [21] that as $N_f \rightarrow \infty$ that they correctly emerge from the three and four point functions of (2.8) involving a quark loop. Therefore to compute fundamental critical exponents in QCD at $O(1/N_f)$ it is sufficient to use the much simpler theory (2.8) to compute these as only one interaction arises. We will point out later where the contributions from 3-gluon vertices would arise in the exponent calculation with respect to perturbation theory. Having defined the NATM in which we will calculate, we define β as

$$\beta = 1 - \eta - \chi \quad (2.9)$$

where χ is the quark gluon vertex anomalous dimension. Also in using a gauge field in a covariant gauge the ghost sector needs to be added to both (2.7) and (2.8). However it turns out that at $O(1/N_f)$ there are no contributions to any exponent we compute and we therefore omit them here, [16].

Finally we state several earlier results which will be required. The β -function of QCD is [2, 23],

$$\begin{aligned} \beta(g) = & (d-4)g + \left[\frac{2}{3}T(R)N_f - \frac{11}{6}C_2(G) \right] g^2 \\ & + \left[\frac{1}{2}C_2(R)T(R)N_f + \frac{5}{6}C_2(G)T(R)N_f - \frac{17}{12}C_2^2(G) \right] g^3 \\ & - \left[\frac{11}{72}C_2(R)T^2(R)N_f^2 + \frac{79}{432}C_2(G)T^2(R)N_f^2 - \frac{205}{288}C_2(R)C_2(G)T(R)N_f \right. \\ & \left. + \frac{1}{16}C_2^2(R)T(R)N_f - \frac{1415}{864}C_2^2(G)T(R)N_f + \frac{2857}{1728}C_2^3(G) \right] g^4 + O(g^5) \end{aligned} \quad (2.10)$$

where the three loop term was given in [24] and the colour group Casimirs are defined as $\text{tr}(T^a T^b) = T(R)\delta^{ab}$, $T^a T^a = C_2(R)$ and $f^{acd}f^{bcd} = \delta^{ab}C_2(G)$. Although in our earlier ansatz we omitted a constant term in the one loop coefficient its contribution to g_c does not appear until $O(1/N_f^2)$ and so

$$g_c = \frac{3\epsilon}{T(R)N_f} + O\left(\frac{1}{N_f^2}\right) \quad (2.11)$$

Various basic exponents are known to $O(1/N_f)$ and we note [15]

$$\eta_1 = \frac{[(2\mu - 1)(\mu - 2) + \mu b]C_2(R)\eta_1^0}{(2\mu - 1)(\mu - 2)T(R)} \quad (2.12)$$

where $\eta = \sum_{i=1}^{\infty} \eta_i(\epsilon)/N_f^i$, $\eta_1^0 = (2\mu - 1)(\mu - 2)\Gamma(2\mu)/[4\Gamma^2(\mu)\Gamma(\mu + 1)\Gamma(2 - \mu)]$ and [16]

$$\chi_1 = - \frac{[(2\mu - 1)(\mu - 2) + \mu b]C_2(R)\eta_1^0}{(2\mu - 1)(\mu - 2)T(R)} - \frac{[(2\mu - 1) + b(\mu - 1)]C_2(G)\eta_1^0}{2(2\mu - 1)(\mu - 2)T(R)} \quad (2.13)$$

Throughout that paper we will work in an arbitrary covariant gauge. We note that the physical operators which occur in the operator product expansion have gauge independent anomalous dimensions and by including a non-zero b this will give us a minor check on the corresponding exponent calculations. The combination $z = A^2 B$ arises too and

$$z_1 = \frac{\Gamma(\mu + 1)\eta_1^0}{2(2\mu - 1)(\mu - 2)T(R)} \quad (2.14)$$

3 Unpolarized operators.

We illustrate the large N_f technique by computing the critical exponent of the simplest operator which arises in the operator product expansion. This is the twist-2 flavour non-singlet operator, [25, 3],

$$\mathcal{O}_{\text{ns},a}^{\mu_1 \dots \mu_n} = i^{n-1} \mathcal{S} \bar{\psi}^I \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} T_{IJ}^a \psi^J - \text{trace terms} \quad (3.1)$$

where \mathcal{S} denotes symmetrization on the Lorentz indices. Although this has already been treated in $1/N_f$ in [14] its value forms part of the flavour singlet calculation detailed later. The full critical exponent associated with (3.1) is $\eta_{\text{ns}}^{(n)}$,

$$\eta_{\text{ns}}^{(n)} = \eta + \eta_{\mathcal{O}} \quad (3.2)$$

The first piece corresponds in exponent language to the wave function renormalization of the constituent fields of (3.1). The second part reflects the renormalization of the operator itself. Although each term of (3.2) is gauge dependent the combination is gauge independent. In perturbation theory the renormalization is carried out by inserting (3.1) in some Green's function and examining its divergence structure. Here we determine the scaling behaviour of the integrals where \mathcal{O}_{ns} is inserted in a quark 2-point function. The two leading order Feynman diagrams are given in fig 1. With the regularization each graph is evaluated with the critical propagators (2.5) in d -dimensions. As in perturbative calculations [3] we project the Lorentz indices of the operator into a basis using a null vector Δ_μ , with $\Delta^2 = 0$. (This is not to be confused with the regularizing parameter Δ which is a scalar object.) They have the general form, omitting the external momentum dependence,

$$\frac{X}{\Delta} + Y + O(\Delta) \quad (3.3)$$

where X and Y are functions of d . The integrals are straightforward to compute using standard rules for massless integrals. To obtain the leading order large N_f contribution α and β are replaced by μ and 1 respectively. Following [19] the residue X of each graph contributes to $\eta_{\mathcal{O},1}^{(n)}$. In this instance we have for the respective graphs

$$\frac{2\mu C_2(R)\eta_1^0}{(\mu - 2)(2\mu - 1)T(R)} \left[1 - b - \frac{(\mu - 1)^3}{(\mu + n - 1)(\mu + n - 2)} \right]$$

and

$$\frac{4\mu(\mu-1)C_2(R)\eta_1^0}{(\mu-2)(2\mu-1)T(R)} \sum_{l=2}^n \frac{1}{(\mu+l-2)} \quad (3.4)$$

where we have included a factor of 2 in the second to account for the contribution of the mirror image and used the value of z , (2.14). Summing the contributions yields, [14],

$$\eta_{\text{ns},1}^{(n)} = \frac{2C_2(R)(\mu-1)^2\eta_1^0}{(2\mu-1)(\mu-2)T(R)} \left[\frac{(n-1)(2\mu+n-2)}{(\mu+n-1)(\mu+n-2)} + \frac{2\mu}{(\mu-1)} [\psi(\mu+n-1) - \psi(\mu)] \right] \quad (3.5)$$

where $\psi(x)$ is the logarithmic derivative of the Γ -function. We recall that this result is in agreement with all known perturbative $\overline{\text{MS}}$ results to three loops, [3, 7]. In concentrating on the detail for this operator we will follow the same procedure in the remainder of the paper with minimal comment.

We now turn to the treatment of the flavour singlet twist-2 operators. Before analysing at the fixed point we need to recall several features of their perturbative renormalization. First, the operators are, [25, 4],

$$\mathcal{O}_F^{\mu_1 \dots \mu_n} = i^{n-1} \mathcal{S} \bar{\psi}^I \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \psi^J - \text{trace terms} \quad (3.6)$$

$$\mathcal{O}_G^{\mu_1 \dots \mu_n} = \frac{1}{2} i^{n-2} \mathcal{S} \text{tr} G^a \mu_1 \nu D^{\mu_2} \dots D^{\mu_{n-1}} G^a_{\nu}{}^{\mu_n} - \text{trace terms} \quad (3.7)$$

As each operator has the same dimension in four dimensions and quantum numbers they mix under renormalization. In other words defining the vector $\mathcal{O}_i = \{\mathcal{O}_F, \mathcal{O}_G\}$ then the bare and renormalized operators are related by

$$\mathcal{O}_i^{\text{ren}} = Z_{ij} \mathcal{O}_j^{\text{bare}} \quad (3.8)$$

where Z_{ij} is a 2×2 matrix of renormalization constants. Consequently the associated anomalous dimension is a 2×2 matrix $\gamma_{ij}(g)$. It has the following structure, with the N_f dependence explicit,

$$\gamma_{ij}(g) = \begin{pmatrix} \gamma^{qq} & \gamma^{gq} \\ \gamma^{qg} & \gamma^{gg} \end{pmatrix} = \begin{pmatrix} a_1 g + (a_2 N_f + a_3) g^2 & b_1 g + (b_2 N_f + b_3) g^2 \\ c_1 N_f g + c_2 N_f g^2 & (d_1 N_f + d_2) g + (d_3 N_f + d_4) g^2 \end{pmatrix} \quad (3.9)$$

where, for example,

$$\begin{aligned} a_1 &= 2C_2(R) \left[4S_1(n) - 3 - \frac{2}{n(n+1)} \right], \quad b_1 = - \frac{4(n^2 + n + 2)C_2(R)}{n(n^2 - 1)} \\ c_1 &= - \frac{8(n^2 + n + 2)T(R)}{n(n+1)(n+2)}, \quad d_1 = \frac{8}{3}T(R) \\ a_2 &= T(R)C_2(R) \left[\frac{4}{3} - \frac{160}{9}S_1(n) + \frac{32}{3}S_2(n) \right. \\ &\quad \left. + \frac{16[11n^7 + 49n^6 + 5n^5 - 329n^4 - 514n^3 - 350n^2 - 240n - 72]}{9n^3(n+1)^3(n+2)^2(n-1)} \right] \\ b_2 &= \frac{32C_2(R)T(R)}{3} \left[\frac{1}{(n+1)^2} + \frac{(n^2 + n + 2)}{n(n^2 - 1)} \left(S_1(n) - \frac{8}{3} \right) \right] \end{aligned} \quad (3.10)$$

The remaining entries of (3.9) can be found in, for instance, [4, 12] and are not important for the present situation. To compute $\gamma_{ij}(g)$ the operators are inserted in both quark and gluon 2-point functions. Various one loop graphs which occur are illustrated in figs 1 and 2. Clearly from (3.9) the N_f dependence is not the same in each term. For example, at g_c the N_f dependence of

each entry is respectively, $O(1/N_f)$, $O(1/N_f)$, $O(1)$ and $O(1)$. Further, in practical applications it is sometimes useful to compute with the operator eigenbasis of (3.9) which simplifies the RGE involving $\gamma_{ij}(g)$ and therefore the evolution of the Wilson coefficients of the operator product expansion. From (3.9) this leads to the eigenvalues

$$\begin{aligned}\lambda_{\pm} &= \frac{1}{2}(d_1 N_f + a_1 + d_2 \pm \sqrt{A_1})g \\ &+ \frac{1}{2}\left((a_2 + d_3)N_f + a_3 + d_4 \pm \frac{A_2}{2\sqrt{A_1}}\right)g^2 + O(g^3)\end{aligned}\quad (3.11)$$

where

$$\begin{aligned}A_1 &= d_1^2 N_f^2 \left[1 + \frac{2(d_4 - a_1)}{d_1 N_f} + \frac{4b_1 c_1}{d_1^2 N_f^2}\right] \\ A_2 &= 2N_f[(d_1(d_3 - a_2) + 2c_1 b_2)N_f \\ &+ (d_2 - a_1)(d_3 - a_2) + d_1(d_4 - a_3) + 2(c_1 b_3 + c_2 b_1)]\end{aligned}\quad (3.12)$$

Or evaluating at g_c the related eigenexponents are at leading order in large N_f

$$\begin{aligned}\lambda_+ &= d_1 N_f g \\ \lambda_- &= \left(a_1 - \frac{b_1 c_1}{d_1}\right)g + \left(a_2 - \frac{b_2 c_1}{d_1}\right)g^2 N_f + O(N_f^2 g^3)\end{aligned}\quad (3.13)$$

Clearly the N_f dependence in each eigenexponent differs. The eigenoperators associated with each eigenvalue, λ_{\pm} , are a combination of the original operators. For example, that associated with λ_- has predominant contributions from the fermionic operator (3.6). Likewise λ_+ is associated primarily with (3.7).

For the critical point analysis there will be a 2×2 matrix of critical exponents analogous to (3.9) which are computed by inserting the critical propagators into the diagrams of figs 1 and 2. In addition the graphs of fig 3 are also of the same order in $1/N_f$. However in determining the contribution to X of each of the graphs it turns out that several are trivial due to the imbalance of the N_f dependence already mentioned. For instance the leading order term for λ_+ arises purely from the tree graph of fig 2. Therefore we take as its entry in $\eta_{ij} \equiv \gamma_{ij}(g_c)$ as [24],

$$\eta_{GG,1} = 2\epsilon \quad (3.14)$$

Also $\eta_{FG,1}$ does not need to be evaluated as its leading order value is given purely by the one loop perturbative result. Next the contribution from the final graph of fig 2 is identically zero. That is, with (2.5) the graph is Δ -finite. Therefore the only non-trivial entry to compute is $\eta_{FF,1}$. As the non-singlet part has already been determined this reduces to evaluating the two loop graphs of fig 3. Each is b -independent and respectively contribute, for even n ,

$$\begin{aligned}& - \frac{\mu(\mu-1)\Gamma(n)\Gamma(2\mu)\eta_1^0}{(\mu-2)(2\mu-1)(\mu+n-1)(\mu+n-2)\Gamma(2\mu-1+n)} \\ & \times [n(n(n-2) + 2(\mu-2+n)(2\mu-3) + (2\mu-2+n)) + 2(n-2)(\mu+n-1)]\end{aligned}$$

and

$$\frac{8\mu(\mu-1)\Gamma(n-1)\Gamma(2\mu)C_2(R)\eta_1^0}{(\mu-2)(2\mu-1)\Gamma(2\mu-1+n)T(R)} \quad (3.15)$$

Hence,

$$\eta_{FF,1}^{(n)} = \frac{(\mu-1)C_2(R)\eta_1^0}{(2\mu-1)(\mu-2)T(R)N_f} \left[\frac{2(\mu-1)(n-1)(2\mu+n-2)}{(\mu+n-1)(\mu+n-2)} + 4\mu[\psi(\mu-1+n) - \psi(\mu)] \right]$$

$$- \frac{\mu\Gamma(n-1)\Gamma(2\mu)}{(\mu+n-1)(\mu+n-2)\Gamma(2\mu-1+n)} \times [(n^2+n+2\mu-2)^2 + 2(\mu-2)(n(n-1)(2\mu-3+2n) + 2(\mu-1+n))] \Big] \quad (3.16)$$

We now discuss the structure of η_{ij} . Unlike the perturbative mixing matrix $\gamma_{ij}(g)$, η_{ij} is triangular. At first sight this would appear to be inconsistent with perturbation theory. However, at leading order in $1/N_f$ the calculation of η_{ij} in fact determines the critical anomalous dimensions of the eigenoperators *directly*. This is not unexpected if one studies the dimensions of (3.6) and (3.7) at g_c . There clearly the canonical dimensions of each operator is different and therefore there is no mixing. The vanishing of certain graphs of fig 2 is merely a reflection of this in the large N_f calculation. This indirect relation between the exponents of the eigenoperators (3.6) and (3.7) is the reason why we distinguish the perturbative entries of (3.9) by q and g in contrast to F and G for the eigenoperators. A further justification of this point of view comes from the comparison of the coefficients of the $O(\epsilon)$ and $O(\epsilon^2)$ terms in the expansion of (3.16) with λ_- evaluated to the same order at g_c . We have checked that they are in total agreement with (3.10) for all n . A further check is that the anomalous dimension must vanish at $n = 2$. Then the original operator corresponds to a conserved physical quantity, the energy momentum tensor which has zero anomalous dimension. From (3.16) it is easy to check that $\eta_{\text{FF},1}^{(2)} = 0$.

It is worth commenting on this calculation in relation to the NATM and QCD equivalence noted earlier, [22]. Clearly λ_- and η_{FF} contain contributions from the insertion of gluonic operators in a Green's function. However the graphs we evaluate to obtain η_{FF} involve only (3.6). The resolution of this apparent inconsistency is obtained by studying the integration of each quark loop in fig 3 with (2.5) and the ϵ expansion of the individual graphs. Clearly in perturbation theory the graphs of fig 1 will contribute to the one loop renormalization whilst those of fig 3 will give part of the two loop value of the anomalous dimension. So one would expect the large N_f graphs to be $O(\epsilon)$ and $O(\epsilon^2)$ respectively. This is not the case. Studying (3.15) each graph of fig 3 is $O(\epsilon)$ and from (3.16) their sum is also of this order. The point is that after performing the quark loop integral and examining the resulting one loop integral, it contains a part which would correspond to the ordinary perturbation theory two loop value as well as a piece that corresponds to the final graph of fig 2 which is a *one* loop integral. In other words an effective gluonic operator like (3.7) emerges naturally in the exponent calculation. In effect we are confirming in our calculation the equivalence observed in [22] where we recall that the three and four point gluon interactions were similarly reproduced by integrating out quark loops.

We conclude this section by giving an indication of the n -dependence of at least the leading order $1/N_f$ coefficients of higher loop terms in the series for $\gamma_-(g)$. Having established the correctness of our expansion at two loops the higher order coefficients are

$$\begin{aligned} a_3 - \frac{b_3 c_1}{d_1} &= \frac{2}{9} S_3(n) - \frac{10}{27} S_2(n) - \frac{2}{27} S_1(n) + \frac{17}{72} - \frac{2(n^2+n+2)^2 [S_2(n) + S_1^2(n)]}{3n^2(n+2)(n+1)^2(n-1)} \\ &\quad - \frac{2S_1(n)(16n^7 + 74n^6 + 181n^5 + 266n^4 + 269n^3 + 230n^2 + 44n - 24)}{9(n+2)^2(n+1)^3(n-1)n^3} \\ &\quad - [100n^{10} + 682n^9 + 2079n^8 + 3377n^7 + 3389n^6 + 3545n^5 + 3130n^4 \\ &\quad + 118n^3 - 940n^2 - 72n + 144] / [27(n+2)^3(n+1)^4 n^4 (n-1)] \end{aligned} \quad (3.17)$$

and

$$a_4 - \frac{b_4 c_1}{d_1} = \frac{2}{27} S_4(n) - \frac{10}{81} S_3(n) - \frac{2}{81} S_2(n) - \frac{2}{81} S_1(n) + \frac{131}{1296}$$

$$\begin{aligned}
& + \zeta(3) \left[\frac{4}{27} S_1(n) - \frac{2}{27n(n+1)} - \frac{1}{9} - \frac{2(n^2+n+2)^2}{9n^2(n+2)(n+1)^2(n-1)} \right] \\
& - \frac{4(n^2+n+2)^2[2S_3(n) + 3S_2(n)S_1(n) + S_1^3(n)]}{27(n+2)n^2(n-1)(n+1)^2} \\
& + 2[S_2(n) + S_1^2(n)] \frac{(16n^7 + 74n^6 + 181n^5 + 266n^4 + 269n^3 + 230n^2 + 44n - 24)}{27n^3(n+2)^2(n+1)^3(n-1)} \\
& - 2S_1(n)[88n^{10} + 608n^9 + 1947n^8 + 3405n^7 + 3670n^6 + 3693n^5 \\
& \quad + 2973n^4 - 8n^3 - 920n^2 - 48n + 144]/[81(n+2)^3(n+1)^4n^4(n-1)] \\
& + [68n^{13} + 548n^{12} + 1861n^{11} + 2474n^{10} - 817n^9 - 4143n^8 - 1712n^7 \\
& \quad - 2871n^6 - 7702n^5 - 2586n^4 + 3136n^3 + 1952n^2 \\
& \quad - 288n - 288]/[81(n+2)^4(n+1)^5(n-1)n^5]
\end{aligned} \tag{3.18}$$

For future reference we list the values of (3.17) calculated for low moments in table 1. A similar table was produced for the analogous coefficient in the non-singlet case. It is important to note that all the fractions up to $n = 8$ are in *exact* agreement with the recent explicit three loop singlet results of [7], when allowance is made for different coupling constant definitions.

4 γ^5 .

To apply the large N_f method to polarized operators we need to review the treatment of γ^5 in perturbation theory and earlier $1/N_f$ calculations. As is well known one must be careful in arbitrary spacetime dimensions when γ^5 or the pseudotensor $\epsilon_{\mu\nu\sigma\rho}$ are present, [26]. The simple reason is that both are purely four dimensional objects unlike, say, γ^μ and $\eta^{\mu\nu}$ and do not generalize in the arbitrary dimensional case. Therefore problems will arise in perturbation theory when one uses dimensional regularization. With this regularization calculations are performed in $d = 4 - 2\epsilon$ dimensions where the infinities are removed before taking the $\epsilon \rightarrow 0$ limit. There are, however, various ways of incorporating γ^5 in such calculations, [26-28]. (A review is, for example, [29].) The original approach of [26] was to split the d -dimensional spacetime into physical and unphysical complements. In the former subspace Lorentz indices run from 1 to 4 whilst they range over the remaining dimensions in the latter. So, for example, the γ -matrices are split into two components

$$\gamma^\mu = \bar{\gamma}^\mu + \hat{\gamma}^\mu \tag{4.1}$$

where the bar, $\bar{}$, denotes the physical four dimensional spacetime and the hat, $\hat{}$, the remaining $(d-4)$ -dimensional subspace. Then the Clifford algebra reduces to

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2\bar{\eta}^{\mu\nu} \quad , \quad \{\bar{\gamma}^\mu, \hat{\gamma}^\nu\} = 0 \quad , \quad \{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\hat{\eta}^{\mu\nu} \tag{4.2}$$

The anti-commutativity of γ^5 is not preserved in the full spacetime. Instead the following relations are used

$$\{\bar{\gamma}^\mu, \gamma^5\} = 0 \quad , \quad [\hat{\gamma}^\mu, \gamma^5] = 0 \tag{4.3}$$

It is known that these definitions give a consistent method for treating γ^5 , [28]. Traces involving an odd number of γ^5 's are performed via, in our conventions,

$$\text{tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho) = 4\epsilon^{\mu\nu\sigma\rho} \tag{4.4}$$

which acts like a projection into the physical dimensions. Further if (4.4) occurs in a loop integral where the γ -matrices are contracted with loop momenta the integral is performed first

and then the Lorentz index contractions carried out, with the caveat that external momenta are physical, $\hat{p}_\mu = 0$.

For high order perturbative calculations this splitting of the algebra is not always practical, [30]. It would be easier if a d -dimensional calculation could be performed. Such an approach has been introduced in [30, 31] and carried out successfully for 3-loop calculations in [30]. The first step there is to replace γ^5 by

$$\gamma^5 = \frac{1}{4!} \epsilon_{\mu\nu\sigma\rho} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho \quad (4.5)$$

and remove the ϵ -tensor from the renormalization procedure. The γ -matrices of (4.5) are treated as d -dimensional in the calculation before projecting to the physical dimension. If two such ϵ -tensors are present then they can be replaced by a sum of products of η -tensors which is treated as d -dimensional. One performs the renormalization in a minimal way as usual to determine the renormalization constants. To complete the calculation, in relation to the $\overline{\text{MS}}$ scheme, one must introduce a finite renormalization constant Z_5 in addition to the first renormalization constant in order to restore the Ward identity, [31].

For the treatment of γ^5 in the $1/N_f$ expansion we recall the simple example of the flavour non-singlet axial current. In [33] the method outlined above was followed to correctly determine the anomalous dimension of $\mathcal{O}_{\text{ns}}^{\mu 5} = \bar{\psi} \gamma^\mu \gamma^5 \psi$ at $O(1/N_f)$. First, if one wishes to find the critical exponent associated with the non-singlet vector current $\mathcal{O}_{\text{ns}}^\mu = \bar{\psi} \gamma^\mu \psi$ then it is inserted in a 2-point function and the residue with respect to Δ is determined. The only relevant graph at $O(1/N_f)$ is the first graph of fig 1. If we insert the more general non-singlet operator $\bar{\psi} \Gamma \psi$ then the contribution to the critical exponent from the graph is

$$- \frac{[\gamma^\nu \gamma^\sigma \Gamma \gamma_\sigma \gamma_\nu - 2\mu(1-b)\Gamma] \eta_1^0}{2(2\mu-1)(\mu-2)T(R)} \quad (4.6)$$

where the square brackets are understood to mean the coefficient of the matrix Γ after all γ -matrix manipulations have been performed for an explicit form of Γ . Therefore for $\Gamma = \gamma^\mu$, (4.6) gives

$$- \frac{[(2\mu-1)(\mu-2) + b\mu] \eta_1^0}{(2\mu-1)(\mu-2)} \quad (4.7)$$

and so with (2.12) and (2.13)

$$\eta_{\mathcal{O}_{\text{ns}}^\mu} = 0 \quad (4.8)$$

consistent with the Ward identity in exponent language, [20]. For $\mathcal{O}_{\text{ns}}^{\mu 5}$ one performs the γ -algebra of (4.6) using (4.2), to give

$$- \frac{[(2\mu-9)(\mu-2) + b\mu] \eta_1^0}{(2\mu-1)(\mu-2)T(R)} \quad (4.9)$$

Thus

$$\tilde{\eta}_{\mathcal{O}_{\text{ns}}^{\mu 5}} = \frac{8\eta_1^0}{(2\mu-1)T(R)} \quad (4.10)$$

where \sim denotes that the object still has to be augmented by the finite renormalization. As discussed in [33] this does not preserve four dimensional chiral symmetry and is not consistent with the Ward identity. To proceed correctly we need to include a finite renormalization constant. In [33] this was computed to be

$$Z_5 = 1 + \frac{C_2(R)\epsilon}{6T(R)N_f} \hat{\text{L}} \left\{ \frac{\ln[1 - 4T(R)N_f a_S/(3\epsilon)]}{B(2-\epsilon, 2-\epsilon)B(3-\epsilon, 1+\epsilon)} \right\} + O\left(\frac{1}{N_f^2}\right) \quad (4.11)$$

where \hat{L} is the Laurent operator which removes non-singular terms from the expansion of the braces and $B(x, y)$ is the Euler β -function. The constant Z_5^{ns} is defined from the requirement that, [30],

$$Z_5^{\text{ns}} \mathcal{R}_{\overline{\text{MS}}} \langle \bar{\psi} \mathcal{O}_{\text{ns}}^{\mu 5} \psi \rangle = \gamma^5 \mathcal{R}_{\overline{\text{MS}}} \langle \bar{\psi} \mathcal{O}_{\text{ns}}^{\mu} \psi \rangle \quad (4.12)$$

where $\mathcal{R}_{\overline{\text{MS}}}$ denotes the R -operator or renormalization procedure. In other words the anti-commutativity of γ^5 is restored by this condition. Using the information in this finite renormalization together with (4.10) the correct $\overline{\text{MS}}$ anomalous dimension does emerge to all orders in the coupling at $O(1/N_f)$.

There are several disadvantages, however, with the form of (4.11). First, it is not as compact as the $O(1/N_f)$ exponents that have been produced in earlier work, [16]. Second by examining (4.11) the result can be simplified since the construction of Z_5^{ns} is in effect equivalent to the difference of the exponents (4.7) and (4.9) at $O(1/N_f)$. In other words the contribution from the finite renormalization to the final $\overline{\text{MS}}$ exponent is equal to

$$- \frac{8\eta_1^0}{(2\mu - 1)T(R)} \quad (4.13)$$

Thus the sum of (4.11) and (4.13) correctly gives in $\overline{\text{MS}}$

$$\eta_{\mathcal{O}_{\text{ns}}^{\mu 5}} = 0 \quad (4.14)$$

Another difficulty with this procedure is that there is a quicker derivation based on features of the fixed point approach. In perturbation theory the regularization used is dimensional in contrast to the critical point method. There the spacetime dimension is fixed and the regularization is analytic as it is the gluon dimension which is adjusted. The upshot is that, at least for non-singlet currents, one can use the anti-commutativity of γ^5 in d -dimensions. Therefore with $\Gamma = \gamma^\mu \gamma^5$ in (4.6) anti-commuting γ^5 twice immediately gives the same contribution as $\Gamma = \gamma^\mu$. Hence the $\overline{\text{MS}}$ result (4.14) follows directly. We have checked this procedure explicitly for other non-singlet operators such as $\bar{\psi} \gamma^5 \psi$ and $\mathcal{S} \bar{\psi} \gamma^5 \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \psi$ by calculating the analogous finite renormalization constant from a condition similar to (4.12) and observing that the result agrees with the direct anti-commuting γ^5 calculation. So, for example, the unpolarized and polarized non-singlet twist-2 operators have the same anomalous dimensions, (3.5). In other words we have justified the use of an anti-commuting γ^5 in non-singlet sectors of calculations. Although much of the content of this section may appear straightforward, there is an important lesson in the result (4.11) from [33] for singlet operators. Then closed quark loops with an odd number of γ^5 matrices will occur which means quantities like (4.11) will need to be computed. As we have demonstrated that this is equivalent to the difference in the anomalous dimensions of the operators of the renormalization condition (4.12) defining the finite renormalization constant, flavour singlet operators can be handled in an efficient way. We will come back to this point in a later section.

5 Polarized singlet operators.

We now extend the unpolarized singlet calculation of section 3 to the polarized case as it is important to compare with recent perturbative calculations [9-11]. The twist-2 operators are [8],

$$\mathcal{O}_F^{\text{pol}} = i^{n-1} \mathcal{S} \bar{\psi} \gamma^5 \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \psi - \text{trace terms} \quad (5.1)$$

$$\mathcal{O}_G^{\text{pol}} = \frac{1}{2} i^{n-2} \mathcal{S} \epsilon^{\mu_1 \alpha \beta \gamma} \text{tr} G_{\beta \gamma} D^{\mu_2} \dots D^{\mu_{n-1}} G^{\mu_n}_{\alpha} - \text{trace terms} \quad (5.2)$$

Several features of the computation of the critical exponents will parallel section 3 such as the triangularity of the mixing matrix and the N_f dependence of $\gamma_{ij}^{\text{pol}}(g)$. The essential difference is the effect γ^5 has in the two two loop graphs of fig 3 which we focus on here. The contribution from the graphs of fig 1 is the same as (3.15).

With (4.4) the second graph of fig 3 is Δ -finite and gives no contribution to $\eta_{\text{FF},1}^{\text{pol}}$. For the other graph one can compute the quark loop in d -dimensions before carrying out the second loop integral, also in arbitrary dimensions. The projection to four dimensions is made at the end. Adding all pieces we have,

$$\begin{aligned} \eta_{\text{FF},1}^{\text{pol}} = & \frac{2C_2(R)\eta_1^0}{(2\mu-1)(\mu-2)T(R)} \left[\frac{(n-1)(2\mu+n-1)(\mu-1)^2}{(\mu+n-1)(\mu+n-2)} \right. \\ & \left. + 2\mu(\mu-1)[\psi(\mu-1+n) - \psi(\mu)] - \frac{\mu(2\mu+n-5)(n+2)\Gamma(n)\Gamma(2\mu)}{2(\mu+n-1)(\mu+n-2)\Gamma(2\mu+n-2)} \right] \end{aligned} \quad (5.3)$$

As in section 3, due to the N_f dependence we have

$$\eta_{\text{GG},1}^{\text{pol}} = 2\epsilon \quad (5.4)$$

We have checked that the ϵ -expansion of (5.3) agrees with the anomalous dimension of the predominantly fermionic eigenoperator of the mixing matrix at two loops, [9-11]. For completeness we note in the notation of (3.9),

$$\begin{aligned} a_1^{\text{pol}} &= 2C_2(R) \left[4S_1(n) - 3 - \frac{2}{n(n+1)} \right] , \quad b_1^{\text{pol}} = - \frac{4(n+2)C_2(R)}{n(n+1)} \\ c_1^{\text{pol}} &= - \frac{8(n-1)T(R)}{n(n+1)} , \quad d_1^{\text{pol}} = \frac{8}{3}T(R) \\ a_2^{\text{pol}} &= T(R)C_2(R) \left[\frac{4}{3} - \frac{160}{9}S_1(n) + \frac{32}{3}S_2(n) + \frac{32[10n^4 + 17n^3 + 10n^2 + 21n + 9]}{9n^3(n+1)^3} \right] \\ b_2^{\text{pol}} &= - \frac{32(n+2)C_2(R)T(R)}{3n(n+1)} \left[S_1(n) - \frac{8}{3} + \frac{(n+2)}{(n+1)} \right] \end{aligned} \quad (5.5)$$

This agreement, moreover, justifies our treatment of γ^5 at the fixed point to be an anti-commuting object whose appearance in closed loops is treated with (4.4). Finally we deduce

$$\begin{aligned} \left[a_3 - \frac{b_3 c_1}{d_1} \right]^{\text{pol}} &= \frac{2}{9}S_3(n) - \frac{10}{27}S_2(n) - \frac{2}{27}S_1(n) + \frac{17}{72} - \frac{2(n+2)(n-1)[S_2(n) + S_1^2(n)]}{3n^2(n+1)^2} \\ &+ \frac{2S_1(n)(13n^3 - 6n^2 + 2n + 3)(n+2)}{9(n+1)^3n^3} \\ &- \frac{[61n^6 + 83n^5 + 27n^4 + 217n^3 + 68n^2 - 36n - 18]}{27(n+1)^4n^4} \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} \left[a_4 - \frac{b_4 c_1}{d_1} \right]^{\text{pol}} &= \frac{2}{27}S_4(n) - \frac{10}{81}S_3(n) - \frac{2}{81}S_2(n) - \frac{2}{81}S_1(n) + \frac{131}{1296} \\ &+ \zeta(3) \left[\frac{4}{27}S_1(n) - \frac{2}{27n(n+1)} - \frac{1}{9} - \frac{2(n+2)(n-1)}{9n^2(n+1)^2} \right] \\ &- \frac{4(n+2)(n-1)[2S_3(n) + 3S_2(n)S_1(n) + S_1^3(n)]}{27n^2(n+1)^2} \end{aligned}$$

$$\begin{aligned}
& + \frac{2(13n^3 - 6n^2 + 2n + 3)(n + 2)[S_2(n) + S_1^2(n)]}{27n^3(n + 1)^3} \\
& - \frac{2S_1(n)(49n^5 - 29n^4 + 95n^3 + 41n^2 - 15n - 9)(n + 2)}{81(n + 1)^4n^4} \\
& + \frac{(19n^8 - 46n^7 + 3n^6 + 195n^5 - 392n^4 - 317n^3 - 32n^2 + 54n + 18)}{81(n + 1)^5n^5}
\end{aligned} \tag{5.7}$$

In the second column of table 1 we have evaluated (5.6) for low moments as a check for future three loop calculations. Although we have given exact fractions for the coefficients at three loops, the numerical values of the polarized and unpolarized entries do not differ significantly as n increases.

6 Singlet axial current.

Having considered a variety of fermionic operators which are both flavour non-singlet and singlet we turn to the remaining current. The renormalization of the singlet axial current $\mathcal{O}_s^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi$ is somewhat special. Unlike the singlet vector current the conservation of $\mathcal{O}_s^{\mu 5}$ is spoiled at the *quantum* level by the chiral anomaly, [34-36]. Consequently under renormalization the composite operator can develop a non-zero anomalous dimension. By contrast the conservation of the vector current ensures it has a zero anomalous dimension at all orders in the coupling constant. Before attacking the problem of computing the $O(1/N_f)$ exponent for $\mathcal{O}_s^{\mu 5}$ in the $\overline{\text{MS}}$ scheme, it is worthwhile reviewing the perturbative approach [32] and in particular [30]. (Other related contributions to the renormalization of the axial anomaly are [37-39].) In the three loop analysis, [30], two renormalization constants are determined in a manner described earlier for other currents. One is the renormalization constant which removes the infinities in the usual way but using the standard γ -algebra and the definition of γ^5 , (4.5). This renormalization does not preserve the axial anomaly, in operator form, in four dimensions. To remedy this a second finite renormalization constant Z_5^{anom} is required. The relevant constraint in the present instance is determined by ensuring that the operator form of the anomaly, [34-36],

$$\partial_\mu \mathcal{O}_s^{\mu 5} = \frac{T(R)N_f}{4g} \epsilon^{\mu\nu\sigma\rho} G_{\mu\nu}^a G_{\sigma\rho}^a \tag{6.1}$$

is preserved, leading to, [36, 29],

$$Z_5^{\text{anom}} \mathcal{R}_{\overline{\text{MS}}} \langle A \partial_\mu \mathcal{O}_s^{\mu 5} A \rangle = \frac{T(R)N_f}{4g} \mathcal{R}_{\overline{\text{MS}}} \langle A \epsilon^{\mu\nu\sigma\rho} G_{\mu\nu}^a G_{\sigma\rho}^a A \rangle \tag{6.2}$$

The large N_f calculation follows this two stage approach. In other words the exponent corresponding to the $\overline{\text{MS}}$ anomalous dimension of $\mathcal{O}_s^{\mu 5}$ is given by

$$\eta_s = \eta + \eta_{5,s} + \eta_5^{\text{fin}} \tag{6.3}$$

It is straightforward to compute the first graph of fig 3 in d -dimensions with the rules given previously. With (4.9)

$$\eta_1 + \eta_{5,s,1} = \frac{C_2(R)\eta_1^0}{T(R)} \left[\frac{8}{(2\mu - 1)} - \frac{6}{(\mu - 1)} \right] \tag{6.4}$$

where the b -dependence has cancelled. We have used a split γ -algebra here to be consistent with the treatment of closed fermion loops in determining η_5^{fin} . There the first graphs of fig 1 and 3

will occur as subgraphs. We have checked that the ϵ -expansion of (6.4) agrees with the three loop result for the same quantity in [30].

To compute η_5^{fin} we use the result of section 4. There with a split γ -algebra the finite renormalization exponent was determined from the difference in the anomalous dimensions of the operators arising in the defining relation. In that case the restoration of the Ward identity was simple in that the result obtained was equivalent to using a fully anti-commuting γ^5 initially and the operators themselves were similar in nature. For η_5^{fin} the exponents of $\partial_\mu \mathcal{O}_s^{\mu 5}$ and $G = \epsilon^{\mu\nu\sigma\rho} G_{\mu\nu}^a G_{\sigma\rho}^a$, which are total derivatives must be determined separately in d -dimensions. In detailing that calculation we focus on $\partial_\mu \mathcal{O}_s^{\mu 5}$ first.

We insert $\partial_\mu \mathcal{O}_s^{\mu 5}$ into a gluon 2-point function as illustrated in fig 4. For the moment we take the momentum flow to be p into the left gluon leg and $(p - q)$ out through the other. This leaves a net flow of q through the operator insertion which is needed since a non-zero momentum must contract with $\gamma^\mu \gamma^5$ in momentum space. To simplify the calculation of each integral we differentiate with respect to q_ϕ and contract with $\epsilon_{\lambda\psi\theta\phi} p^\theta$ where λ and ψ are the Lorentz indices of the gluon legs. Then q is set to zero, [30]. This procedure ensures that part of the integrals contributing to the renormalization of the operator is projected out. We have given the $O(1/N_f)$ diagrams in figs 5 and 6. The former is the one loop anomaly and with the critical propagators it is Δ -finite. However, as the remaining graphs represent the higher order corrections the value of the first graph of fig 5 must be factored off each to leave a formal sum of terms

$$- \frac{6T(R)N_f(2\mu - 1)(\mu - 2)}{(\mu - 1)} \left[1 + \frac{1}{N_f} \left(\frac{X}{\Delta} + O(1) \right) \right] \quad (6.5)$$

The overall factor is the d -dimensional value of the anomaly which is non-zero in four dimensions. In (6.5) we have included z_1 from the amplitudes of the quark fields which explains the origin of the factor $(\mu - 2)$. The residue X is the value of the $O(1/N_f)$ part of the dimension of $\partial_\mu \mathcal{O}_s^{\mu 5}$ we require.

With the momentum flow as indicated we have computed the value of each graph of fig 6. No graphs have been included where the vertex with the external gluon is dressed. These graphs together with the vertex counterterm do not contribute to X as they are Δ -finite in sum. With the critical propagators only the first two graphs are non-zero and give

$$X = - \frac{C_2(R)\eta_1^0}{T(R)} \left[\frac{[(2\mu - 9)(\mu - 2) + b\mu]}{(2\mu - 1)(\mu - 2)} + \frac{3}{(\mu - 1)} \right] \quad (6.6)$$

The remaining graphs are each Δ -finite and we note the colour factors of the last two graphs are each $C_2(G)$. Recalling the field content of $\partial_\mu \mathcal{O}_s^{\mu 5}$ we have

$$\eta_{\partial\mathcal{O},1} = - \frac{C_2(R)\eta_1^0}{T(R)} \left[\frac{8}{(2\mu - 1)} - \frac{3}{(\mu - 1)} \right] \quad (6.7)$$

The treatment of G is parallel to that just outlined. With the same projection of momenta the tree graph of fig 5 gives the normalization value of (-6) analogous to that of (6.5). The relevant graphs are given in fig 7 and we list their respective contributions to X as

$$\begin{aligned} & - \frac{C_2(R)\eta_1^0}{T(R)} \quad , \quad - \frac{[2C_2(R) - C_2(G)]\eta_1^0}{T(R)} \quad , \quad \frac{C_2(G)[4\mu^2 - 6\mu + 1 + b]\eta_1^0}{2(2\mu - 1)(\mu - 2)T(R)} \\ & - \frac{C_2(G)[8\mu^2 - 13\mu + 4 - \mu(1 - b)]\eta_1^0}{2(2\mu - 1)(\mu - 2)T(R)} \end{aligned} \quad (6.8)$$

Useful in carrying out this calculation was the symbolic manipulation programme FORM, [40]. The value of the three loop graph accounted for the most tedious part of the calculation. However, we made use in part of results of integrals which arose in the computation of the QCD β -function, [41]. This was achieved by computing the dimension of the composite operator $(G_{\mu\nu}^a)^2$ associated with the coupling constant in a gluon 2-point function. We have included a non-zero b to observe its cancellation as a minor calculational check. Although the graphs involved in computing the dimension of G in fig 7 are similar in topology to those for $\partial\mathcal{O}_s^{\mu_5}$ in fig 6, the values obtained are somewhat different. For example, the last graphs of each figure are similar once the loop integral with the singlet current insertion is performed which leaves a Feynman integral with an effective G insertion. The difference in the values arises due to the critical propagators used and the fact that this loop integral changes the dimension of the gluon lines contracted with it and therefore the nature of the remaining loop integrations. One check on this is that the leading terms in the ϵ expansion of each graph ought to agree. It is easy to observe that the first two values of (6.8) give the same leading coefficient as the second term of (6.6). Likewise the remaining two terms of (6.8) are $O(\epsilon)$.

With the field content dimension (2.12) and (2.13), we find

$$\eta_{G,1} = - \frac{3C_2(R)\eta_1^0}{T(R)} \quad (6.9)$$

It is reassuring to note the cancellation of the terms involving $C_2(G)$ again as the overall $\overline{\text{MS}}$ renormalization of $\mathcal{O}_s^{\mu_5}$ at $O(1/N_f)$ is expected to be proportional to $C_2(R)$ only.

With (6.7) and (6.9) the finite renormalization exponent is

$$\eta_{5,1}^{\text{fin}} = - \frac{C_2(R)\eta_1^0}{T(R)} \left[\frac{8}{(2\mu-1)} + \frac{3(\mu-2)}{(\mu-1)} \right] \quad (6.10)$$

Therefore

$$\eta_{s,1} = - \frac{3\mu C_2(R)\eta_1^0}{(\mu-1)T(R)} \quad (6.11)$$

where the cancellation of the terms proportional to $8/(2\mu-1)$ reflects the non-singlet calculation of section 4. A final check on this relatively simple result is that it correctly reproduces the large N_f leading order two and three loop $\overline{\text{MS}}$ coefficients of [30, 32]. This agreement, moreover, again strengthens the validity of our treatment of γ^5 . Consequently we deduce, in the notation of (2.1) and our coupling constant conventions,

$$a_4 = - \frac{4C_2(R)}{27} \quad , \quad a_5 = \frac{[9\zeta(3) - 7]C_2(R)}{81} \quad (6.12)$$

7 Discussion.

We conclude our study by remarking on possible future calculations in this area. The natural task to be performed next will be the $O(1/N_f^2)$ corrections to the non-singlet twist-2 operators. Such a calculation would mimic the determination of the mass operator dimension but would require the quark dimension η_2 first. Only the abelian values are available for both these quantities, [21]. On another front the corrections to (3.14) and (5.4) are needed. This would parallel the calculation of the QCD β -function in $1/N_f$, [41]. Both these results for the gluonic operators would give important insight into the n -dependence of the higher order anomalous dimensions and the x -behaviour of the DGLAP splitting functions.

From a more mathematical physics point of view such analyses may become important for studying the operator content of strictly four dimensional gauge theories which have (infrared) fixed points, [42, 43]. Evaluating the perturbative anomalous dimension of the composite operator at these points would be necessary to gain information on the (conformal) field content of the underlying theory in the perturbatively accessible region. Moreover the existence of fixed points such as that of Banks and Zaks in QCD, [42], for a range of N_f values have been the subject of recent interest in supersymmetric theories with various gauge groups and matter content, [43]. Therefore any information that can be determined from traditional field theory methods and which sum perturbation theory beyond present low orders such as $1/N_f$, could be used to compare estimates of, for example, critical exponents deduced from exact non-perturbative arguments.

Acknowledgements. The author acknowledges support for this work through a PPARC Advanced Fellowship, thanks Dr D.J. Broadhurst for encouragement and Drs A. Vogt and J. Blümlein for useful discussion on their work. The tedious algebra was performed in part through use of FORM, [39], and REDUCE, [44].

A DGLAP splitting functions.

In this appendix we discuss the relation of our results to the DGLAP splitting functions, $P(x, g)$. The x -dependence of the higher order contributions to these functions is currently of interest in relation to low x physics. We recall that the anomalous dimensions of the twist-2 operators are related to the DGLAP splitting functions via a Mellin transform with the proviso that x , which is the variable conjugate to the moment n , is restricted to the unit interval. First, we note that our convention is

$$\int_0^1 dx x^{n-1} P(x, g) = \frac{1}{4} \gamma^{(n)}(g) \quad (\text{A.1})$$

Of course (A.1) can be evaluated at the fixed point and a critical exponent which sums the leading order in $1/N_f$ of the splitting function can be deduced. So, for example, the twist-2 non-singlet anomalous dimension (3.5) gives the result,

$$\begin{aligned} P_{\text{ns},1}(x, g_c) = & \frac{2C_2(R)(\mu-1)^2\eta_1^0}{(2\mu-1)(\mu-2)T(R)} \left[\frac{(\mu^2-4\mu+1)}{(\mu-1)^2} \delta(1-x) \right. \\ & \left. - \mu(\mu-1)x^{\mu-2}(1-x) + \frac{2\mu x^{\mu-2}}{(\mu-1)} - \frac{2\mu x^{\mu-2}}{(\mu-1)(1-x)_+} \right] \quad (\text{A.2}) \end{aligned}$$

where we use the standard notation in the final term to ensure sensible behaviour in the $x \rightarrow 1$ limit. We have checked that the ϵ -expansion of (A.2) correctly reproduces the $O(1/N_f)$ part of the two loop non-singlet splitting function of [3]. In light of this we determine the three loop structure as

$$a_3^{\text{ns}} = \frac{17}{72} \delta(1-x) - \frac{(1+x^2) \ln^2 x}{18(1-x)} - \frac{(11x^2-12x+11) \ln x}{27(1-x)} - \frac{2(7-6x)}{27} + \frac{2}{27(1-x)_+} \quad (\text{A.3})$$

The treatment of the unpolarized and polarized singlet cases are similar but we detail only the latter case as the exponent has a simpler n -dependence. Moreover recent articles have examined the polarized splitting functions in the small x limit, [46]. Performing the inverse

Mellin transform the splitting function becomes

$$\begin{aligned}
P_{s,1}^{\text{pol}}(x, g_c) = & \frac{2C_2(R)\mu(\mu-1)^2\eta_1^0}{(2\mu-1)(\mu-2)T(R)} \left[\frac{(\mu^2-4\mu+1)}{\mu(\mu-1)^2} \delta(1-x) - (\mu-1)x^{\mu-2}(1-x) \right. \\
& + \frac{2x^{\mu-2}}{(\mu-1)} - \frac{2x^{\mu-2}}{(\mu-1)(1-x)_+} - \frac{\mu(2\mu-1)x^{\mu-2}(1-x)}{\mu(\mu-1)} \\
& \left. - \frac{(2\mu-1)(\mu-3)(\mu-4)x^{\mu-2}}{(\mu-1)} [xB_{1-x}(2\mu-2, 1-\mu) - B_{1-x}(2\mu-2, 2-\mu)] \right]
\end{aligned} \tag{A.4}$$

where $B_x(p, q)$ is the incomplete β -function which has the integral representation

$$B_x(p, q) = \int_0^x du u^{p-1} (1-u)^{q-1} \tag{A.5}$$

We can once again perform the ϵ -expansion of (A.4) and attempt to compare with the explicit two loop results of [10, 11, 9]. We find from (A.4)

$$\left[a_1 - \frac{b_1 c_1}{d_1} \right]^{\text{pol}} = -\frac{3}{2} \delta(1-x) - 6(1+x) \ln x - \frac{(8-15x+8x^2)}{(1-x)} \tag{A.6}$$

$$\begin{aligned}
\left[a_2 - \frac{b_2 c_1}{d_1} \right]^{\text{pol}} = & \frac{1}{12} \delta(1-x) + (1+x) \ln^2 x + 10(1-x) \ln(1-x) + \frac{2(6-5x^2) \ln x}{3(1-x)} \\
& - 4(1+x) [\text{Li}_2(x) - \text{Li}_2(1)] + \frac{2(13-21x+13x^2)}{9(1-x)}
\end{aligned} \tag{A.7}$$

where the dilogarithm function $\text{Li}_2(x)$ enters, [45]. Its appearance, however, would seem to suggest that (A.7) does not relate to information in the two loop splitting matrix since $\text{Li}_2(x)$ is absent at leading order in $1/N_f$ there, [10, 11]. (It does occur, for example, at next to leading order in $1/N_f$ in all entries bar γ^{qq} .) The resolution of this rests in the triangularity property of the critical point mixing matrix. In order to correctly compare with the explicit perturbative matrix it has to be mapped to a similar structure. This is achieved by an invertible 2×2 matrix R . In other words, in matrix language,

$$P_s^{\text{tri}}(x, g_c) = R \left(\int_0^1 dx x^{n-1} P_s^{\text{pert}}(x, g_c) \right) R^{-1} \tag{A.8}$$

As we are only interested in the $O(1/N_f)$ FF component this reduces to comparing (A.6) and (A.7) with the sum of the products of the appropriate elements of (A.8) in x -space which is a combination of entries similar to the left side of (A.6). Also since R is n -dependent we need to express this product of n -dependent functions as a Mellin transform of a single x -dependent function for the comparison. Useful in this respect is the convolution formula for the Mellin transform. As splitting functions are defined to be zero outside the unit interval this takes the following form in this instance

$$\mathcal{M}[f_1(x)] \mathcal{M}[f_2(x)] = \mathcal{M} \left[\int_x^1 \frac{du}{u} f_1 \left(\frac{x}{u} \right) f_2(u) \right] \tag{A.9}$$

where the Mellin transform is defined to be

$$\mathcal{M}[f(x)] = \int_0^1 dx x^{n-1} f(x) \tag{A.10}$$

So with (A.9) and, at leading order,

$$\begin{aligned}
a_1^{\text{pol}} &= C_2(R) \left[1 + x - \frac{2}{(1-x)_+} - \frac{3}{2}\delta(1-x) \right] , \quad b_1^{\text{pol}} = - (2-x)C_2(R) \\
c_1^{\text{pol}} &= - 2(2x-1)T(R) , \quad d_1^{\text{pol}} = \frac{2T(R)}{3}\delta(1-x) \\
a_2^{\text{pol}} &= C_2(R)T(R) \left[\frac{1}{12}\delta(1-x) + \frac{(11-12x+11x^2)}{9(1-x)_+} + \frac{(1+x^2)}{2(1-x)}\ln x \right. \\
&\quad \left. - (1-x) + (1-3x)\ln x + (1+x)\ln^2 x \right] \\
b_2^{\text{pol}} &= 4C_2(R)T(R) \left[\frac{1}{9}(x+4) + \frac{1}{3}(2-x)\ln(1-x) \right]
\end{aligned} \tag{A.11}$$

it is straightforward to verify that (A.6) and (A.7) emerge.

Finally, having established the relation of (A.4) with perturbation theory it is a simple exercise to produce

$$\begin{aligned}
\left[a_3 - \frac{b_3 c_1}{d_1} \right]^{\text{pol}} &= \frac{17}{72}\delta(1-x) - \frac{4}{3}(1+x)\ln x \ln^2(1-x) - \frac{10}{3}(1-x)\ln^2(1-x) \\
&\quad - \frac{4}{3}\ln x \ln(1-x) - \frac{(1+x)}{9}\ln^3 x - \frac{(6-5x^2)}{9(1-x)}\ln^2 x \\
&\quad + \frac{4}{3}(1+x)\ln x [\text{Li}_2(x) - \text{Li}_2(1)] - \frac{10}{9}(1-x)\ln(1-x) \\
&\quad + \frac{2(8-9x)(2+3x)}{27(1-x)}\ln x + \frac{2}{9}(5+11x)[\text{Li}_2(x) - \text{Li}_2(1)] \\
&\quad + \frac{4}{3}(1+x)[\text{Li}_3(x) - \text{Li}_3(1) - \ln x \text{Li}_2(x)] - \frac{4}{3}\text{Li}_2(1-x) \\
&\quad + \frac{8}{3}(1+x)[\text{Li}_3(1-x) - \ln(1-x)\text{Li}_2(1-x)] + \frac{(55-108x+55x^2)}{27(1-x)_+}
\end{aligned} \tag{A.12}$$

In relation to the work of [46] we deduce from (A.6), (A.7) and (A.12) that the leading small x behaviour of these perturbative coefficients are, respectively,

$$- 6\ln x , \quad \ln^2 x , \quad - \frac{1}{9}\ln^3 x \tag{A.13}$$

In respect of the large N_f and small x limits, [46] concludes that these do not commute. Although this may seem to be a disappointing result it is worth recalling that the primary motivation of this paper is the provision of information on higher order coefficients of operator dimensions which can be compared with explicit perturbative calculations.

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n	$a_3 - b_3 c_1/d_1$	$[a_3 - b_3 c_1/d_1]^{\text{pol}}$
2	0	$-\frac{7}{108}$
4	$-\frac{121259}{720000}$	$-\frac{3496333}{19440000}$
6	$-\frac{3166907}{13891500}$	$-\frac{26205857}{111132000}$
8	$-\frac{1328467729}{5038848000}$	$-\frac{466437737839}{1728324864000}$
10	$-\frac{304337312935261}{1054350180576000}$	$-\frac{309708615382541}{1054350180576000}$
12	$-\frac{842357166098254633}{2737572318857376000}$	$-\frac{853943993349679513}{2737572318857376000}$
14	$-\frac{42512567719680559}{131614053791220000}$	$-\frac{343857572061363287}{1052912430329760000}$
16	$-\frac{755896148277147625515451}{2251271656795440254976000}$	$-\frac{762758545067630140156811}{2251271656795440254976000}$
18	$-\frac{1121815282809553973842772849}{3235896767484248927963904000}$	$-\frac{1130341459796219185897273169}{3235896767484248927963904000}$
20	$-\frac{78640886458671664340562623}{220772683941420492161280000}$	$-\frac{26382991363083553371777301}{73590894647140164053760000}$
22	$-\frac{4248342909129791924572989157741}{11650178316263341224273468364800}$	$-\frac{4271977516708367936099843954621}{11650178316263341224273468364800}$

Table 1. $O(1/N_f)$ coefficients for unpolarized and polarized fermionic twist-2 singlet operators at three loops as a function of moment n .

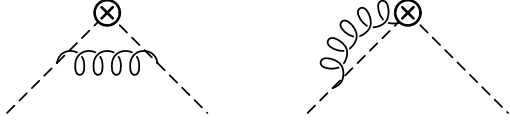


Fig. 1. Leading order graphs for $\eta_{\text{ns}}^{(n)}$.

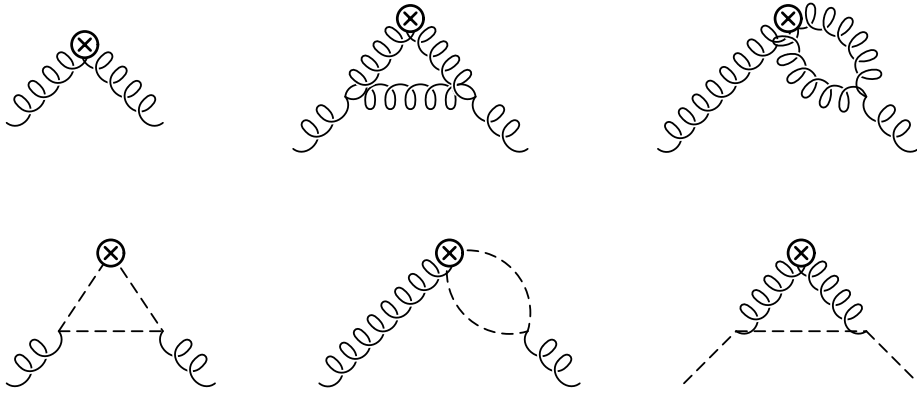


Fig. 2. One loop graphs for singlet operators.

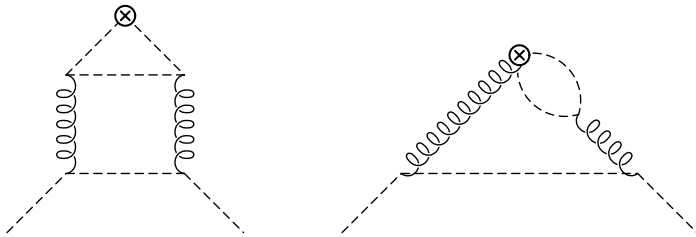


Fig. 3. Additional graphs for singlet operators.



Fig. 4. Operator insertions in gluon 2-point functions.



Fig. 5. Leading order graphs for $\partial_\mu \mathcal{O}_s^{\mu 5}$ and $\epsilon^{\mu\nu\sigma\rho} G_{\mu\nu}^a G_{\sigma\rho}^a$ insertions.

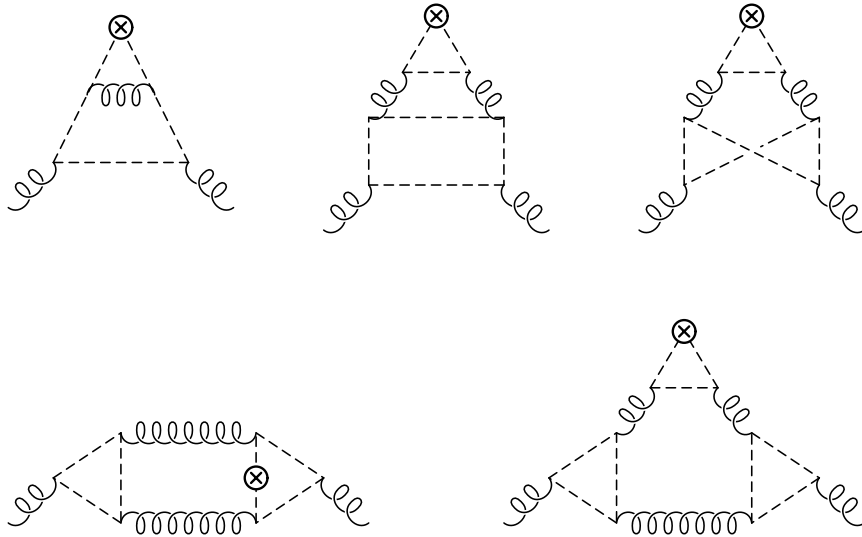


Fig. 6. Graphs for $\partial_\mu \mathcal{O}_s^{\mu 5}$ insertion.

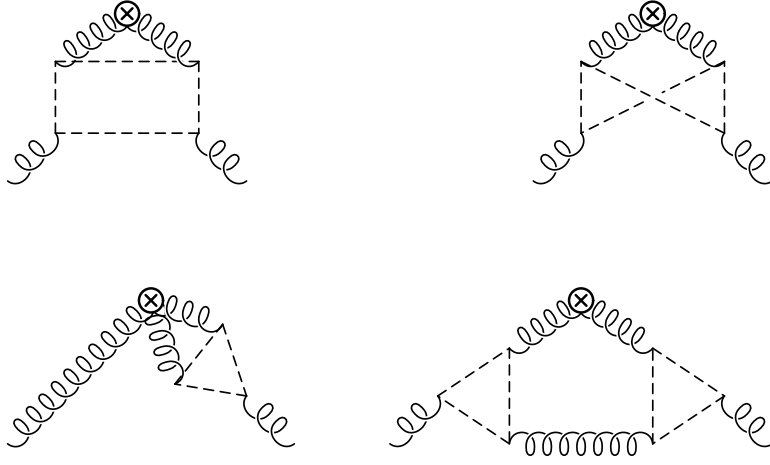


Fig. 7. Graphs for $\epsilon^{\mu\nu\sigma\rho} G_{\mu\nu}^a G_{\sigma\rho}^a$ insertion.